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Average Width and Optimal Interpolation of the Sobolev–Wiener Class $W^r_{pq}(\mathbf{R})$ in the Metric $L_q(\mathbf{R})$

FANG GENSUN*,† AND LIU YONGPING*

Department of Mathematics, Beijing Normal University, Beijing 100875, People's Republic of China

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In this paper, we determine the exact value of average $n-K$ width $\bar{d}_n(W^r_{pq}(\mathbf{R}), L_q(\mathbf{R}))$ of Sobolev–Wiener class $W^r_{pq}(\mathbf{R})$ in the metric $L_q(\mathbf{R})$ for $1<q\leqslant p<\infty$ and get the value of $\bar{d}_n(W^r_p(R), L_{qp}(R))$ for the dual case. We also solve the optimal interpolation problems of $W^r_{pq}(\mathbf{R})$ in the metric $L_q(\mathbf{R})$ and $W^r_p(\mathbf{R})$ in the metric $L_{qp}(\mathbf{R})$ for $1<q\leqslant p<\infty$.

1. INTRODUCTION

Let X be a normed linear space. For subsets A and B of X we define

$$E(A, B, X) =: \sup_{f\in A} \inf_{g\in B} \|f-g\|_X.$$

We denote by $X(\mathbf{R})$ a normed linear space of functions defined on the entire real axis $\mathbf{R}$ and assume that $f\in X(\mathbf{R})$ implies $f|_{I_N}\in X(\mathbf{R})$ for any $N>0$, where $f|_{I_N}(x)=f(x)$ or 0 according to whether $x\in I_N =: [-N, N]$ or not. All functions $f|_{I_N}$ in $X(R)$ compose a subspace of $X(R)$ which we denote by $X(I_N)$.

For $A\subset X(R)$, $N>0$, and $\varepsilon>0$, we define

$$S(A)_N =: \{f|_{I_N}(x) : \|f\|_{X(\mathbf{R})}\leqslant 1, f\in A\}$$

and

$$K(\varepsilon, N, A) =: \min\{\dim B\colon B\subset X(I_N), E(S(A)_N, B, X(I_N))<\varepsilon\},$$

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† Visiting Scholar at Mathematisches Institut of Universität Erlangen–Nürnberg in Germany during 1990–1992.

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where $\dim B$ denotes the dimension of the finite dimensional linear subspace B of $X(I_N)$. It is easy to see that $K(\varepsilon, N, A)$ is non-decreasing in N and non-increasing in ε.

Let σ be a positive real number. A linear subspace A of $X(R)$ is said to be of average dimension σ if

$$\overline{\dim}\, A =: \lim_{\varepsilon \to 0} \varliminf_{N \to \infty} \frac{K(\varepsilon, N, A)}{2N} = \sigma < \infty. \tag{1.1}$$

For a subset $\mathfrak{M}$ of $X(\mathbf{R})$, the quantity

$$\bar{d}_\sigma(\mathfrak{M}, X(\mathbf{R})) =: \inf\{E(\mathfrak{M}, A, X(\mathbf{R})) \colon \overline{\dim}\, A \leqslant \sigma, A \subset X(\mathbf{R})\} \tag{1.2}$$

is called the Kolmogorov average σ-width (briefly average $\sigma - K$ width) of the set $\mathfrak{M}$ in $X(\mathbf{R})$. If there is a subspace A^* of $X(\mathbf{R})$ of average dimension $\overline{\dim}\, A^* \leqslant \sigma$ such that

$$\bar{d}_\sigma(\mathfrak{M}, X(\mathbf{R})) = E(\mathfrak{M}, A^*, X(\mathbf{R})), \tag{1.3}$$

then A^* is called optimal for $\bar{d}_\sigma(\mathfrak{M}, X(\mathbf{R}))$.

The concept of average width was first proposed by Tikhomirov [19, 20] in order to consider problems of optimal approximation methods on a non-compact Sobolev set $W_p^r(\mathbf{R})$. Sun Yongsheng [16] proposed problems of optimal interpolation on $W_p^r(\mathbf{R})$. For $p = q$ many results have been obtained on $W_p^r(\mathbf{R})$ in the metric $L_p(\mathbf{R})$ (see [3, 8, 12, 13, 16]). It is easy to verify that $\bar{d}_n(W_p^r(\mathbf{R}), L_q(\mathbf{R})) = \infty$ for $p > q$, $n \in \mathbf{Z}_+ = \{1, 2, ...\}$. For $q = 1 \leqslant p \leqslant \infty$, or $1 \leqslant q \leqslant p = \infty$, Liu Yongping and Sun Yongsheng [9–11] considered the average $n - K$ width of the set $W_{pq}^r(R)$ in $L_q(\mathbf{R})$, where $W_{pq}^r(\mathbf{R})$ $(1 \leqslant p, q \leqslant \infty)$ is defined as follows:

For each $r \in \mathbf{Z}_+$, set

$$\begin{aligned} L_{pq}^r(\mathbf{R}) =: \{f \in L_q(\mathbf{R}) \colon f^{(r-1)} &\text{ is abs. cont.} \\ &\text{on every finite interval}, f^{(r)} \in L_{pq}(\mathbf{R})\}, \end{aligned} \tag{1.4}$$

$$W_{pq}^r(\mathbf{R}) =: \{f \in L_{pq}^r(\mathbf{R}) \colon \|f^{(r)}\|_{pq} \leqslant 1\},$$

where $L_{pq}(\mathbf{R}) =: \{f \colon \|f\|_{pq} < \infty\}$ and

$$\|g\|_{pq} = \begin{cases} \{\sum_{j \in \mathbf{Z}} \|g(\cdot + j)\|_{L_p[0,1]}^q\}^{1/q}, & 1 \leqslant q < \infty \\ \sup_{j \in \mathbf{Z}} \|g(\cdot + j)\|_{L_p[0,1]}, & q = \infty, \end{cases} \tag{1.5}$$

while $\|\cdot\|_{L_p(I)}$ denotes the usual L_p-norm on the interval I, and $\mathbf{Z} =: \{0, \pm 1, \pm 2, ...\}$.

In [5], the following properties of $L_{pq}(\mathbf{R})$ were shown:

(i) $L_{pq}(\mathbf{R})$ is a Banach space with norm $\|\cdot\|_{pq}$ for any $1 \leqslant p,\ q \leqslant \infty$.

(ii) If $p \geqslant q$, then $L_{pq}(\mathbf{R}) \subset L_p(\mathbf{R}) \cap L_q(\mathbf{R})$; if $p \leqslant q$, then $L_{pq}(\mathbf{R}) \supset L_p(\mathbf{R}) \cup L_q(\mathbf{R})$; if $p = q$, $L_{pq}(\mathbf{R}) = L_p(\mathbf{R})$.

$W^r_{pq}(\mathbf{R})$ is called the Sobolev–Wiener class. When $p = q$, $W^r_{pq}(\mathbf{R}) = W^r_p(\mathbf{R})$ is the usual Sobolev class. For convenience, we often write W^r_{pq}, L_{pq}, and $\|\cdot\|_p$ instead of $W^r_{pq}(\mathbf{R})$, $L_{pq}(\mathbf{R})$, and $\|\cdot\|_{pp}$, respectively.

In the following, we always assume

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{q} + \frac{1}{q'} = 1 \qquad (1 \leqslant p, q \leqslant \infty).$$

The main purpose of this paper is to determine the exact value of average $n-K$ width $\bar{d}_n(W^r_{pq}, L_q)$ of Sobolev–Wiener class W^r_{pq} in the metric L_q for $1 < q \leqslant p < \infty$. For the dual case we determine the value of $\bar{d}_n(W^r_p, L_{qp})$. We also solve the optimal interpolation problem of W^r_{pq} in the metric L_q, and W^r_p in the metric L_{qp} for $1 < q \leqslant p < \infty$.

Before stating our main results, we give two lemmas as follows:

LEMMA 1.1 (cf. [1]). *Let $1 < q \leqslant p < \infty$. Then there are two* 1*-periodic functions $\phi(x)$ and $\psi(x)$ such that*

(1) $\phi(x) = \int_0^1 D_r(x-t)\,|\psi(t)|^{p'-1} \operatorname{sgn} \psi(t)\, dt$,
$\psi(t) = \lambda^{-q} \int_0^1 D_r(x-t)\,|\phi(x)|^{q-1} \operatorname{sgn} \phi(x)\, dx$
$(\lambda =: \lambda(r, p, q))$;

(2) *$\phi(x)$ has only two simple zeros* 0 *and* $\frac{1}{2}$ *in* $[0, 1)$, *$\psi(x)$ has only two simple zeros α_r and $\frac{1}{2} + \alpha_r$, where*

$$\alpha_r = \tfrac{1}{8}\,(1 + (-1)^{r+1}); \tag{1.6}$$

(3) $\phi(x + \frac{1}{2}) = -\phi(x)$, $\psi(x + \frac{1}{2}) = -\psi(x)$;

(4) $\|\phi\|_{L_q[0,1]} = \lambda(r, p, q)$, *and* $\|\phi^{(r)}(\cdot)\|_{L_p[0,1]} = 1$; (1.7)

(5) $\lambda(r, p, q) = \lambda(r, q', p')$. (1.8)

Here

$$D_r(t) = 2(2\pi)^{-r} \sum_{k=1}^{\infty} k^{-r} \cos(2k\pi t - \tfrac{1}{2}\pi r)$$

is the 1*-periodic Bernoulli polynomial.*

Let

$$\Phi_n(x)=\left(\frac{2}{n}\right)^r \phi\left(\frac{n}{2}x\right), \qquad \Psi_n(x)=\left(\frac{2}{n}\right)^r \psi\left(\frac{n}{2}x\right), \qquad n\in \mathbf{Z}_+. \quad (1.9)$$

Then $\Phi_n(x)$ and $\Psi_n(x)$ are two $(2/n)$-periodic functions which depend only on r, n, p, and q. For $n\in\mathbf{Z}_+$, let

$$S_{n,r-1}=\Big\{s(t)\in C^{r-2}\colon s^{(r)}(t)=0,$$
$$\forall t\in\left(\frac{1}{n}(j+2\alpha_r), \frac{1}{n}(j+1+2\alpha_r)\right), \forall j\in\mathbf{Z}\Big\}$$

be the subspace of polynomial splines of degree $r-1$ with simple knots $\{(j+2\alpha_r)/n\}_{j\in\mathbf{Z}}$. Any polynomial spline $s(x)$ of $S_{1,r-1}$ is called a cardinal spline function.

LEMMA 1.2 (cf. [15]). *If there are two constants $c>0$, $\beta>0$ such that $|f(x)|\leqslant c(|x|^\beta+1)$, then there is a unique interpolation operator $s_{r-1}(f,x)\in S_{1,r-1}$ such that*

$$s_{r-1}(f,j)=f(j)\ \forall j\in\mathbf{Z} \qquad \text{and} \qquad |s_{r-1}(f,x)|=O|x|^\beta) \quad (x\to\infty).$$

Our main results are as follows.

THEOREM 1. *Let $1<q\leqslant p<\infty$, $r\in\mathbf{Z}_+$. Then*

$$\begin{aligned}\bar{d}_n(W^r_{pq}, L_q)&=E(W^r_{pq}, S_{n,r-1}, L_q)\\ &=\sup_{f\in W^r_p}\|f(\cdot)-s_{n,r-1}(f,\cdot)\|_q=n^{-r}\tilde{\lambda}(r,p,q),\end{aligned} \quad (1.10)$$

where $s_{n,r-1}(f,x)=: s_{r-1}(f(\cdot/n), nx)$, $\tilde{\lambda}(r,p,q)=: \|\Phi_1(\cdot)\|_{L_q[0,1]}=2^r\lambda(r,p,q)$ and $n\in\mathbf{Z}_+$. For $\sigma\geqslant 1$, the strong asymptotic result

$$\bar{d}_\sigma(W^r_{pq}, L_q)=\sigma^{-r}\tilde{\lambda}(r,p,q)+o(\sigma^{-r}) \quad (1.11)$$

holds.

Remark 1.1. For some special cases, (1.10) was solved in [12] ($p=q\in\{1,2,\infty\}$), [10] ($1\leqslant q\leqslant p=\infty$, $1=q\leqslant p\leqslant\infty$), and [3, 13] ($1\leqslant q=p\leqslant\infty$), respectively.

In Sections 2 and 3, we give the upper and lower estimates for $\bar{d}_n(W^r_{pq}, L_q)$, $1\leqslant q\leqslant p\leqslant\infty$, respectively. In Section 3, we also obtain a result analogous to Theorem 1 on infinite dimensional widths [8] of W^r_{pq}

in L_q. In Section 4 we consider optimal interpolation of W^r_{pq} in L_q. Finally, in Section 5, we consider the dual case and obtain the following results:

THEOREM 2. *Let* $1 < q \leqslant p < \infty$, $r \in \mathbf{Z}_+$, $n \in \mathbf{Z}_+$. *Then*

$$\begin{aligned}\bar{d}_n(W^r_p, L_{qp}) &= E(W^r_p, S_{n,r-1}, L_{qp}) \\ &= \sup_{f \in W^r_p} \|f(\cdot) - s_{n,r-1}(f, \cdot)\|_{qp} \\ &= n^{-r}\tilde{\lambda}(r, q', p') = n^{-r}\tilde{\lambda}(r, p, q).\end{aligned}$$

From Theorems 1 and 2, we have

THEOREM 3. *Let* $1 < q \leqslant p < \infty$, $n \in \mathbf{Z}_+$. *Then*

$$\begin{aligned}\bar{d}_n(W^r_{pq}, L_q) &= \bar{d}_n(W^r_{q'p'}, L_{p'}) = \bar{d}_n(W^r_p, L_{qp}) \\ &= \bar{d}_n(W^r_{q'}, L_{p'q'}) = \sup_{f \in W^r_{pq}} \|f(\cdot) - s_{n,r-1}(f, \cdot)\|_q \\ &= \sup_{f \in W^r_p} \|f(\cdot) - s_{n,r-1}(f, \cdot)\|_{qp} = n^{-r}\tilde{\lambda}(r, p, q).\end{aligned}$$

2. AN UPPER BOUND FOR $\bar{d}_n(W^r_{pq}, L_q)$

LEMMA 2.1 (cf. [4, 14, 15]). *Let* $L_r(x) \in S_{1,r-1}$ *be the cardinal spline satisfying* $L_r(v) = \delta_{0v}$, *where* $\delta_{0v} = 0$ *for any* $v \in \mathbf{Z}\backslash\{0\}$ *and* $\delta_{00} = 1$, *and* $L_r(x) = O(e^{-\beta_0|x|})$, $|x| \to \infty$, *for some* $\beta_0 > 0$. *Then, for any* $f \in W^r_{pq}$, *the cardinal spline interpolation formula with remainder*

$$f(x) - s_{r-1}(f, x) = \int_{\mathbf{R}} K_r(x, t)\, f^{(r)}(t)\, dt \qquad (\forall x \in \mathbf{R})$$

holds. Here $K_r(x, t)$ *possesses the following properties*:

(1) $K_r(x, t) = (1/(r-1)!)((x-t)^{r-1}_+ - \sum_{v \in \mathbf{Z}} (v-t)^{r-1}_+ L_r(x-v))$;

(2) $K_r(x+1, t+1) = K_r(x, t)$ $(\forall x, t \in \mathbf{R})$;

(3) $|K_r(x, t)| \leqslant Ae^{-\beta_0|x-t|}$ $(\forall x, t \in \mathbf{R})$;

(4) $\operatorname{sgn} K_r(x, t) = (-1)^{[r/2]} \operatorname{sgn} \sin \pi x \operatorname{sgn} \sin \pi(t - 2\alpha_r)$ $(r \geqslant 2, \forall x, t \in \mathbf{R})$;

(5) $K_r(x, t) = (-1)^r K_r(t - 2\alpha_r, x - 2\alpha_r)$ $(\forall x, t \in \mathbf{R})$.

LEMMA 2.2 (cf.[6]; Jensen's inequality). *If $f(t)\geqslant 0$ and $\int_{\mathbf{R}} f(t)\,dt=1$, then for any $g(t)\in L_\infty(\mathbf{R})$, we have*

$$\int_{\mathbf{R}} f(t)\,|g(t)|\,dt\leqslant\left\{\int_{\mathbf{R}} f(t)\,|g(t)|^q\,dt\right\}^{1/q}\qquad (1\leqslant q<\infty). \tag{2.1}$$

By virtue of Lemma 2.1 and by changing scale, we have

$$f(x)-s_{\sigma,r-1}(f,x)=\int_{\mathbf{R}} K_{\sigma r}(x,t)\,f^{(r)}(t)\,dt, \tag{2.2}$$

where $K_{\sigma r}(x,t)=:\sigma^{-r+1}K_r(\sigma x,\sigma t)$, and $s_{\sigma,r-1}(f,x)=:s_{r-1}(f(\cdot/\sigma),\sigma x)$, for any positive real number σ.

We prove that for any $n\in\mathbf{Z}_+$ the inequality

$$\sup_{f\in W^r_{pq}(\mathbf{R})}\|f(\cdot)-s_{n,r-1}(f,\cdot)\|_q\leqslant n^{-r}\tilde{\lambda}(r,p,q) \tag{2.3}$$

holds.

Let $f\in W^r_{pq}(\mathbf{R})$. Then, by Lemmas 1.1 and 1.2, we have

$$\int_{\mathbf{R}} K_{nr}(x,t)\,\Phi_n^{(r)}(t)(\Phi_n(x))^{-1}\,dt=1,$$

$$K_{nr}(x,t)\,\Phi_n^{(r)}(t)\,(\Phi_n(x))^{-1}\geqslant 0, \tag{2.4}$$

$$\begin{aligned}\|f(\cdot)-s_{n,r-1}(f,\cdot)\|_q &=\left\{\int_{\mathbf{R}}\left|\int_{\mathbf{R}} K_{nr}(x,t)\,f^{(r)}(t)\,dt\right|^q dx\right\}^{1/q}\\ &=\left\{\int_{\mathbf{R}}\left|\int_{\mathbf{R}}\frac{K_{nr}(x,t)\,\Phi_n^{(r)}(t)}{\Phi_n(x)}\cdot\frac{f^{(r)}(t)}{\Phi_n^{(r)}(t)}\,dt\right|^q |\Phi_n(x)|^q\,dx\right\}^{1/q}.\end{aligned} \tag{2.5}$$

By Jensen's inequality, we get that for all $x\in\mathbf{R}$,

$$\begin{aligned}&\left|\int_{\mathbf{R}}\frac{K_{nr}(x,t)\,\Phi_n^{(r)}(t)}{\Phi_n(x)}\cdot\left|\frac{f^{(r)}(t)}{\Phi_n^{(r)}(t)}\right|dt\right|^q\\ &\qquad\leqslant\int_{\mathbf{R}}\frac{K_{nr}(x,t)\,\Phi_n^{(r)}(t)}{\Phi_n(x)}\cdot\left|\frac{f^{(r)}(t)}{\Phi_n^{(r)}(t)}\right|^q dt.\end{aligned} \tag{2.6}$$

By (2.4)–(2.6) and the Fubini theorem, we have

$$\begin{aligned}&\|f(\cdot)-s_{n,r-1}(f,\cdot)\|_q^q\\ &\quad\leqslant\int_{\mathbf{R}}\left\{\int_{\mathbf{R}} K_{nr}(x,t)\,|\Phi_n(x)|^{q-1}\operatorname{sgn}\Phi_n(x)\,dx\right\}\\ &\qquad\times|f^{(r)}(t)|^q\,|\Phi_n^{(r)}(t)|^{1-q}\operatorname{sgn}\Phi_n^{(r)}(t)\,dt.\end{aligned} \tag{2.7}$$

According to Lemma 1.1 and the definitions of $\Phi_n(x)$ and $\Psi_n(x)$, we get

$$\Psi_n^{(r)}(t) = (-1)^r \lambda^{-q} \left(\frac{n}{2}\right)^{r(q-1)} |\Phi_n(t)|^{q-1} \operatorname{sgn} \Phi_n(t),$$

$$\Psi_n(t) = \left(\frac{2}{n}\right)^r |\Phi_n^{(r)}(t)|^{p-1} \operatorname{sgn} \Phi_n^{(r)}(t). \tag{2.8}$$

On the other hand, by Lemmas 2.1 and 2.2, we see that

$$\Psi_n(t) = (-1)^r \int_{\mathbf{R}} K_{nr}(x, t)\, \Psi_n^{(r)}(x)\, dx. \tag{2.9}$$

Therefore, from (2.8) and (2.9), we have

$$\begin{aligned}
&\int_{\mathbf{R}} K_{nr}(x, t) |\Phi_n(x)|^{q-1} \operatorname{sgn} \Phi_n(x)\, dx \\
&\quad = (-1)^r \lambda^q \left(\frac{2}{n}\right)^{r(q-1)} \int_{\mathbf{R}} K_{nr}(x, t)\, \Psi_n^{(r)}(x)\, dx \\
&\quad = \left(\frac{2}{n}\right)^{r(q-1)} \lambda^q \Psi_n(t) \\
&\quad = \left(\frac{2}{n}\right)^{rq} \lambda^q |\Phi_n^{(r)}(t)|^{p-1} \operatorname{sgn} \Phi_n^{(r)}(t).
\end{aligned} \tag{2.10}$$

From (2.7) and (2.10), we have

$$\|f(\cdot) - s_{n,r-1}(f, \cdot)\|_q^q \leqslant \left(\frac{2}{n}\right)^{rq} \lambda^q \int_{\mathbf{R}} |f^{(r)}(t)|^q\, |\Phi_n^{(r)}(t)|^{p-q}\, dt. \tag{2.11}$$

Since $f \in W_{pq}^r(\mathbf{R})$, if $p = q$, then

$$\int_{\mathbf{R}} |f^{(r)}(t)|^q |\Phi_n^{(r)}(t)|^{p-q}\, dt = \int_{\mathbf{R}} |f^{(r)}(t)|^q\, dt \leqslant 1. \tag{2.12}$$

If $p > q$, let $s = p/q$, $1/s + 1/s' = 1$. By Hölder's inequality we have

$$\begin{aligned}
&\int_0^1 |\Phi_n^{(r)}(t)|^{p-q} |f^{(r)}(t+j)|^q\, dt \\
&\quad \leqslant \left(\int_0^1 |\Phi_n^{(r)}(t)|^{(p-q)s'}\, dt\right)^{1/s'} \left(\int_0^1 |f^{(r)}(t+j)|^{qs}\, dt\right)^{1/s}, \quad \forall j \in \mathbf{Z}.
\end{aligned}$$

Since $s' = p(p-q)^{-1}$ and $\int_0^1 |\Phi_n^{(r)}(t)|^p\, dt = \int_0^1 |\phi^{(r)}(t)|^p\, dt = 1$, we have

$$\int_0^1 |\Phi_n^{(r)}(t)|^{p-q} |f^{(r)}(t+j)|^q\, dt \leqslant \left(\int_0^1 |f^{(r)}(t+j)|^p\, dt\right)^{q/p}. \tag{2.13}$$

Therefore, by (2.13) we have

$$\begin{aligned}\int_{\mathbf{R}} &|\Phi_n^{(r)}(t)|^{p-q} |f^{(r)}(t)|^q\, dt \\ &= \sum_{j\in\mathbf{Z}} \int_0^1 |\Phi_n^{(r)}(t)|^{p-q} |f^{(r)}(t+j)|^q\, dt \\ &\leqslant \sum_{j\in\mathbf{Z}} \|f^{(r)}(\cdot+j)\|^q_{L_p[0,1]} = \|f^{(r)}\|^q_{pq} \leqslant 1. \end{aligned} \tag{2.14}$$

Thus, from (2.11)–(2.14), we obtain

$$\|f(\cdot) - s_{n,r-1}(f,\cdot)\|_q \leqslant \left(\frac{2}{n}\right)^r \lambda(r,p,q) = n^{-r}\bar{\lambda}(r,p,q) \qquad (n\in\mathbf{Z}_+), \tag{2.15}$$

which is (2.3). Since $\dim(S_{n,r-1}|_{I_N}) \leqslant (2N)n + r$, it is easy to see that $\overline{\dim}\, S_{n,r-1} \leqslant n$. Therefore, we have

$$\bar{d}_n(W^r_{pq}, L_q) \leqslant \sup_{f\in W^r_{pq}} \|f(\cdot) - s_{n,r-1}(f,\cdot)\|_q \leqslant n^{-r}\bar{\lambda}(r,p,q). \tag{2.16}$$

Thus, the upper estimate for $\bar{d}_n(W^r_{pq}, L_q)$ is complete.

3. Proof of Theorem 1

Let

$$\begin{aligned}\tilde{W}^r_p[a,b] = \{f : f^{(r-1)} &\text{ is abs. cont. on } [a,b],\ f^{(j)}(a) = f^{(j)}(b), \\ &j = 0, 1, ..., r-1,\ \|f^{(r)}\|_{L_p[a,b]} \leqslant 1\}.\end{aligned}$$

Then $\tilde{W}^r_p[a,b]$ is the Sobolev class of functions with period $b-a$. Put

$$W^{r,0}_p[a,b] = \{f\in \tilde{W}^r_p[a,b] : f^{(j)}(a) = 0,\ j = 0, 1, ..., r-1\}, \tag{3.1}$$

$$F_{pq}(M_n, [a,\ b]) = \{f\in W^{r,0}_p[a,b] : E(f, M_n, L_q[a,b]) = \|f\|_{L_q[a,b]}\}, \tag{3.2}$$

where M_n denotes a subspace of $L_q[a,b]$ of dimension n for $n\in\mathbf{Z}_+$.

LEMMA 3.1. *Let $n \in \mathbf{Z}_+$ and $1 < q \leqslant p < \infty$. If M_n is a linear subspace of $L_q[0, 1]$ of dimension n, then*

(1) $\sup\{\|f\|_{L_q[0,1]}; f \in F_{pq}(M_n, [0, 1])\} \geqslant d_{n+r+2}(\tilde{W}_p^r[0,1], L_q[0,1])$,

(2) $d_n(W_p^{r,0}[0, 1], L_q[0, 1]) \geqslant d_{n+r+2}(\tilde{W}_p^r[0, 1], L_q[0, 1])$,

(3) (cf. [1]) $d_{2m}(\tilde{W}_p^r[0, 1], L_q[0, 1]) = m^{-r}\lambda(r, p, q)$ $(m \in \mathbf{Z}_+,\ 1 \leqslant q \leqslant p \leqslant +\infty)$,

where the quantity

$$d_n(\mathfrak{M}, X) = \inf_{H_n} \sup_{f \in \mathfrak{M}} \inf_{g \in H_n} \|f - g\|_X,$$

in which the H_n range over all linear subspaces of X of dimension at most n, is the Kolmogorov n-width of $\mathfrak{M}$ in X.

Proof. To prove assertion (1) of Lemma 3.1 we make use of Buslaev's method (see [1, 2]). Let $S^{n+r+1} = \{\xi \in \mathbf{R}^{n+r+2}: \sum_{i=1}^{n+r+2} |\xi_i| = 1\}$. For $\xi = (\xi_1, ..., \xi_{n+r+2}) \in S^{n+r+1}$, set $t_0 = 0$, $t_k = \sum_{i=1}^{k} |\xi_i|$ $(k = 1, 2, ..., n+r+2)$. If $t_j > t_{j-1}$, define $h_0(t, \xi) = \operatorname{sgn} \xi_k$, $t \in (t_{k-1}, t_k)$. Otherwise, we let $h_0(t, \xi) = 0$. Put

$$f_0(x, \xi) = D_r * h_0(\cdot, \xi)(x) + \beta_0 =: \int_0^1 D_r(x-t)\, h_0(t, \xi)\, dt + \beta_0, \quad (3.3)$$

where β_0 is taken such that $\inf_{c \in \mathbf{R}} \|D_r * h_0(\cdot, \xi) + c\|_{L_q[0,1]} = \|D_r * h_0(\cdot, \xi) + \beta_0\|_{L_q[0,1]}$.

Let

$$f_k(x, \xi) = \int_0^1 D_r(x-t)\, h_k(t, \xi)\, dt + \beta_k, \quad (3.4)$$

where $h_k(t, \xi)$ and β_{k+1} satisfy the conditions

$$\int_0^1 D_r(x-t)\, |f_k(x, \xi)|^{q-1} \operatorname{sgn} f_k(x, \xi)\, dx = \lambda_{k+1}^q\, |h_{k+1}(t, \xi)|^{p-1} \operatorname{sgn} h_{k+1}(t, \xi),$$

$$\inf_{c \in \mathbf{R}} \|D_r * h_{k+1}(\cdot, \xi) + c\|_{L_q[0,1]} = \|D_r * h_{k+1}(\cdot, \xi) + \beta_{k+1}\|_{L_q[0,1]}.$$

Here $\lambda_k = \lambda_k(r, p, q)$ is taken such that

$$\|h_{k+1}(\cdot, \xi)\|_{L_p[0,1]} = 1.$$

Let $M_n = \operatorname{span}\{g_1, ..., g_n\} \subset L_q[0, 1]$, $\dim(M_n) = n$, $\xi \in S^{n+r+1}$. From the strict convexity of $L_q[0, 1]$ $(1 < q < \infty)$, we know that $f_k(x, \xi)$ has a unique best approximation $\sum_{i=1}^{n} c_i g_i(x)$ by the subspace M_n in $L_q[0, 1]$. Put

$$\eta_j(\xi) = c_j \ (j = 1, ..., n), \qquad \eta_{n+j+1}(\xi) = f_k^{(j)}(0, \xi) \ (j = 0, ..., r-1),$$

$$\eta_{n+r+1}(\xi) = \int_0^1 h_0(t, \xi)\, dt.$$

It is easy to verify that $\eta(\xi) =: (\eta_1(\xi), ..., \eta_{n+r+1}(\xi))$ is a continuous and odd mapping from S^{n+r+1} into $\mathbf{R}^{n+r+1}$. Then, by Borsuk's theorem (cf. [7]), there is a point $\xi_0 \in S^{n+r+1}$ such that $\eta(\xi_0) = 0$. Therefore, we have

$$\begin{aligned}\|f_k(\cdot, \xi_0)\|_{L_q[0,1]} &= \inf_{\{a_i\}} \left\| f_k(\cdot, \xi_0) - \sum_{i=1}^{n} a_i g_i(\cdot) \right\|_{L_q[0,1]} \\ &= E(f_k(\cdot, \xi_0), M_n, L_q[0, 1]).\end{aligned}$$

Since $f_k(x, \xi_0)$ is a 1-periodic function and

$$\|f_k^{(r)}(\cdot, \xi_0)\|_{L_p[0,1]} = \|h_k(\cdot, \xi_0)\|_{L_p[0,1]} = 1,$$

then $f_k^{(j)}(0, \xi_0) = f_k^{(j)}(1, \xi_0) = 0$, $j = 0, 1, ..., r-1$, and hence we have $f_k(x, \xi_0) \in W_p^{r,0}[0, 1]$, and $f_k(x, \xi_0) \in F_{pq}(M_n, [0, 1])$.

Let $m = [(n + r + 2)/2] + 1$. Then we have

$$\begin{aligned}\|f_k(\cdot, \xi_0)\|_{L_q[0,1]} &\geq \min_{\xi \in S^{2m}} \|f_k(\cdot, \xi)\|_{L_q[0,1]}, \\ \sup\{\|f\|_{L_q[0,1]} : f \in F_{pq}(M_n, [0, 1])\} &\geq \min_{\xi \in S^{2m}} \|f_k(\cdot, \xi)\|_{L_q[0,1]}.\end{aligned} \tag{3.5}$$

By [1], we have

$$\begin{aligned}&\lim_{k \to \infty} \min_{\xi \in S^{2m}} \|f_k(\cdot, \xi)\|_{L_q[0,1]} \\ &\quad = \min_{\xi \in S^{2m}} \lim_{k \to \infty} \|f_k(\cdot, \xi)\|_{L_q[0,1]} \geq d_{2m}(\tilde{W}_p^r[0, 1], L_q[0, 1]) \\ &\quad \geq d_{n+r+2}(\tilde{W}_p^r[0, 1], L_q[0, 1]).\end{aligned} \tag{3.6}$$

Therefore, from (3.5) and (3.6), we obtain (1) of Lemma 3.1. Part (2) of Lemma 3.1 follows from (1) of Lemma 3.1.

Proof of Theorem 1. For $\sigma > 0$, let M be a subspace of $L_q(R)$ of average dimension $\overline{\dim}\, M \leq \sigma$, and B be a subspace of $L_q(I_N)$, $N > 0$, satisfying

$$N_\sigma =: \dim B = K(\varepsilon, N, M) \qquad \text{and} \qquad E(S(M)_N, B, L_q(I_N)) < \varepsilon.$$

For each $f \in F_{pq}(B, I_N)$ and $g \in M$, $\|g\|_q \leqslant 2\|f|_{I_N}\|_q$, we have

$$\begin{aligned}\|f-g\|_{L_q(I_N)} &\geqslant \inf_{h\in B}\|f-h\|_{L_q(I_N)} - \inf_{h\in B}\|g-h\|_{L_q(I_N)}\\ &\geqslant \|f\|_{L_q(I_N)} - 2\|f\|_{L_q(I_N)}\, E(S(M)_N, B, L_q(I_N))\\ &\leqslant (1-2\varepsilon)\|f\|_{L_q(I_N)} = (1-2\varepsilon)\|f|_{I_N}\|_{L_q(\mathbf{R})}.\end{aligned} \tag{3.7}$$

Therefore, for each $f \in F_{pq}(B, I_N)$, we have

$$\inf\{\|f|_{I_N} - g\|_q : g \in M\} \geqslant (1-2\varepsilon)\|f|_{I_N}\|_q. \tag{3.8}$$

It is easy to verify that for any $f \in W_p^{r,0}(I_N)$, $f|_{I_N} \in (2N)^{1/q-1/p} W_{pq}^r$, $1 \leqslant q \leqslant p \leqslant \infty$. Then, from (3.8) we get

$$\begin{aligned}&E(W_{pq}^r, M, L_q)\\ &\quad\geqslant (2N)^{1/p-1/q} E(W_p^{r,0}(I_N), M|_{I_N}, L_q(I_N))\\ &\quad\geqslant (2N)^{1/p-1/q} E(F_{pq}(B, I_N), M|_{I_N}, L_q(I_N))\\ &\quad\geqslant (1-2\varepsilon)(2N)^{1/p-1/q}\sup\{\|f\|_{L_q(I_N)} : f \in F_{pq}(B, I_N)\}.\end{aligned} \tag{3.9}$$

On the other hand, by changing scale and (3) of Lemma 3.1, we have

$$\begin{aligned}&d_{N_\sigma+r+2}(W_p^r(I_N), L_q(I_N))\\ &\quad= (2N)^{r+1/q-1/p} d_{N_\sigma+r+2}(\tilde{W}_p^r[0,1], L_q[0,1])\\ &\quad\geqslant (2N)^{r+1/q-1/p}\left(\left[\frac{N_\sigma+r+2}{2}\right]+1\right)^{-r}\lambda(r,p,q).\end{aligned} \tag{3.10}$$

Combining (3.9), (2) of Lemma 3.1, and (3.10), we have

$$E(W_{pq}^r, M, L_q) \geqslant (1-2\varepsilon)(2N)^r\left(\left[\frac{N_\sigma+r+2}{2}\right]+1\right)^{-r}\lambda(r,p,q). \tag{3.11}$$

By the definition of N_σ, we have

$$\lim_{\varepsilon\to 0}\ \varliminf_{N\to\infty}\frac{N_\sigma}{2N} \leqslant \sigma.$$

Further, for any subspace M of L_q of average dimension $\overline{\dim}\, M \leqslant \sigma$, by (3.11) we have

$$E(W_{pq}^r, M, L_q) \geqslant \sigma^{-r} 2^r \lambda(r,p,q) = \sigma^{-r}\tilde{\lambda}(r,p,q). \tag{3.12}$$

Thus, we obtain

$$\bar{d}_\sigma(W^r_{pq}, L_q) \geqslant \sigma^{-r}\tilde{\lambda}(r, p, q) \qquad (\forall \sigma > 0). \tag{3.13}$$

Hence, (1.10) follows (2.16) and (3.13) for $\sigma = n \in \mathbf{Z}_+$.

If $\sigma > 1$, $1 < q \leqslant p < \infty$, then we may choose an integer m such that $m < \sigma < m+1$. Further, we have

$$\bar{d}_{m+1}(W^r_{pq}, L_q) \leqslant \bar{d}_\sigma(W^r_{pq}, L_q) \leqslant \bar{d}_m(W^r_{pq}, L_q).$$

Hence, by (1.10) and (3.13), (1.11) is immediately obtained. The proof of Theorem 1 is complete.

In [8], Li Chun proposed the following concept of infinite dimensional width.

Let X be a normed linear space of functions defined on $\mathbf{R}$, M be a linear subspace of X. For each $\sigma > 0$, if $\underline{\lim}_{N\to\infty} (2N)^{-1} \dim(M|_{[-N,N]}) = \sigma$, then we say that the dimensional index of M is σ, and denote it by $\widetilde{\dim}\, M = \sigma$. Let $\mathfrak{M}$ be a subset of X. The quantity

$$\tilde{d}_\sigma(\mathfrak{M}, X) = \inf_{\widetilde{\dim}\, M \leqslant \sigma} \sup_{f \in \mathfrak{M}} \inf_{g \in M} \|f - g\|_X$$

is called the infinite dimensional $\sigma - K$ width of $\mathfrak{M}$ in X.

LEMMA 3.2.

$$\bar{d}_\sigma(\mathfrak{M}, X) \leqslant \tilde{d}_\sigma(\mathfrak{M}, X). \tag{3.14}$$

Proof. Let M be a linear subspace of X. Then for all $\varepsilon > 0$,

$$E(S(M)_N, M|_{I_N}, X(I_N)) = 0 < \varepsilon.$$

If $\widetilde{\dim}\, M \leqslant \sigma$ $(\sigma > 0)$, then $\underline{\lim}_{N\to+\infty} (2N)^{-1} \dim(M|_{I_N}) \leqslant \sigma$, i.e., for all $\eta > 0$, there is a real number $N(\eta) > 0$ such that $\dim(M|_{I_N}) \leqslant 2N(\sigma + \eta)$ holds for any $N > N(\eta)$. Thus, from the definition of $K(\varepsilon, N, M)$, we know that when $N > N(\eta)$,

$$K(\varepsilon, N, M) \leqslant \dim(M|_{I_N}) \leqslant (\sigma + \eta)\, 2N, \qquad \forall \eta > 0.$$

Therefore we have

$$\overline{\dim}\, M = \lim_{\varepsilon\to 0} \underline{\lim}_{N\to\infty} \frac{K(\varepsilon, N, M)}{2N} \leqslant \sigma.$$

By the definition of $\tilde{d}_\sigma(\mathfrak{M}, X)$ and $\bar{d}_\sigma(\mathfrak{M}, X)$, we have (3.14).

THEOREM 4. *Let* $1 < q \leqslant p < \infty$. *Then*

$$(1) \quad \tilde{d}_n(W^r_{pq}, L_q) = \bar{d}_n(W^r_{pq}, L_q) = n^{-r}\tilde{\lambda}(r, p, q) \qquad (\text{if } n \in \mathbf{Z}_+). \quad (3.15)$$

$$(2) \quad \tilde{d}_\sigma(W^r_{pq}, L_q) = \sigma^{-r}\tilde{\lambda}(r, p, q) + o(\sigma^{-r}) \qquad (\text{if } \sigma \geqslant 1). \quad (3.16)$$

Proof. From (3.13), (3.14), we have

$$\tilde{d}_n(W^r_{pq}, L_q) \geqslant n^{-r}\tilde{\lambda}(r, p, q) \qquad (\text{if } \sigma = n \in \mathbf{Z}_+). \quad (3.17)$$

Since $\underline{\lim}_{N \to \infty} (2N)^{-1} \dim(S_{n,r-1}|_{I_N}) \leqslant \underline{\lim}_{N \to \infty} (2N)^{-1}(2Nn + r) \leqslant n$, from (2.16) and the definition of $\tilde{d}_n(W^r_{pq}, L_q)$, we have

$$\tilde{d}_n(W^r_{pq}, L_q) \leqslant n^{-r}\tilde{\lambda}(r, p, q) \qquad (\text{if } n \in \mathbf{Z}_+). \quad (3.18)$$

Hence, (3.15) follows (3.17) and (3.18). Thus, by (3.13)–(3.15), we obtain (3.16).

Remark 3.1. If $1 < p = q < \infty$, $\sigma > 0$, by changing scale, we obtain

$$\sup_{f \in W^r_p} \|f(\cdot) - s_{\sigma, r-1}(f, \cdot)\|_p = \sigma^{-r}\tilde{\lambda}(r, p, p).$$

Here $s_{\sigma, r-1}(f, \cdot)$ is the interpolation operator of splines defined in (2.2). Therefore in this case we obtain the exact estimations of $\tilde{d}_\sigma(W^r_p, L_p)$ and $\bar{d}_\sigma(W^r_p, L_p)$ for any real number $\sigma > 0$.

4. Optimal Interpolation of W^r_{pq} in L_q.

In many recent books (cf. [18, 21, 22]), the function classes on which the optimal recovery problems were investigated are defined on a compact set, for example, on a bounded closed interval, on the unit circle, or on the unit disk of the complex plane. In [16], Sun Yongsheng proposed and discussed the optimal interpolation problem on some classes of differentiable functions defined on the entire real axis. Following [16], we denote by Θ_σ the set of sequences $\xi = \{\xi_j\}_{j \in \mathbf{Z}}$ satisfying the conditions

$$\xi_j < \xi_{j+1}, \forall j \in \mathbf{Z}, \qquad \text{and} \qquad \underline{\lim}_{N \to +\infty} \frac{\operatorname{card}(\xi \cap [-N, N])}{2N} \leqslant \sigma, \quad (4.1)$$

where $\sigma > 0$ is fixed and $\operatorname{card}(\xi \cap [-N, N])$ is the number of ξ_j contained in $[-N, N]$. Given $\xi \in \Theta_\sigma$, $\xi = \{\xi_j\}_{j \in \mathbf{Z}}$, and $f \in C(\mathbf{R})$, the set $I(\xi) = \{f(\xi)\}_{j \in \mathbf{Z}}$ is called a sample (or information) operator of $f(x)$. For the solution operator $S = I$ (identity operator), the diameter of information and the minimal information diameter (cf. [22]) on $\mathfrak{M} \subset X$ are defined as, respectively,

$$D_{\xi,\sigma}(\mathfrak{M}, X) = \sup\{\|f_1 - f_2\|_X : f_1, f_2 \in \mathfrak{M}, I_\xi(f_1) = I_\xi(f_2)\}, \tag{4.2}$$

$$D_\sigma(\mathfrak{M}, X) = \inf_{\xi \in \Theta_\sigma} D_{\xi,\sigma}(\mathfrak{M}, X). \tag{4.3}$$

Here X is a normed linear space of function with norm $\|\cdot\|_X$. For fixed $\xi \in \Theta_\sigma$, let $\phi: I_\xi(\mathfrak{M}) \to X$ be a mapping which may be taken as an algorithm for the solution operator I (i.e., interpolation problem) on $\mathfrak{M}$ in X. The optimal intrinsic error on $\mathfrak{M}$ of the solution operator I in X defined by

$$E_\sigma(\mathfrak{M}, X) = \inf_{\xi \in \Theta_\sigma} \inf_{\phi} \sup_{f \in \mathfrak{M}} \|f - \phi(I_\xi(f))\|_X. \tag{4.4}$$

When the algorithms ϕ of (4.4) run only over the set of linear mappings defined on a linear set Y $(\supset I_\xi(\mathfrak{M}))$, then we arrive at the optimal linear intrinsic error which is denoted by $E_\sigma^L(\mathfrak{M}, X)$. From [18], if $\mathfrak{M}$ is symmetric about its center, then

$$\tfrac{1}{2} D_\sigma(\mathfrak{M}, X) \leqslant E_\sigma(\mathfrak{M}, X) \leqslant E_\sigma^L(\mathfrak{M}, X). \tag{4.5}$$

LEMMA 4.1. (cf. [11, 17]). *Let* $1 < q \leqslant p \leqslant \infty$. *Then*

$$E(W_{q'}^r, S_{r-1}(\xi) \cap L_{p'q'}, L_{p'q'}) \leqslant \sup\{\|f\|_q : f \in W_{pq}^r, f(\xi_j) = 0, \forall j \in \mathbf{Z}\},$$

where $S_{r-1}(\xi) = \{s(t) \in C^{r-2}(R); s^{(r)}(t) = 0, \forall t \in (\xi_j, \xi_{j+1}), \forall j \in \mathbf{Z}\}$.

THEOREM 5. *Let* $1 < q \leqslant p < \infty$.

(1) $\frac{1}{2} D_n(W_{pq}^r, L_q) = E_n(W_{pq}^r, L_q) = E_n^L(W_{pq}^r, L_q) = \sup_{f \in W_{pq}^r} \|f(\cdot) - s_{n,r-1}(f, \cdot)\|_q = d_n(W_{pq}^r, L_q) = n^{-r}\tilde{\lambda}(r, p, q)$ (*if* $n \in \mathbf{Z}_+$).

(2) $\sigma^{-r}\tilde{\lambda}(r, p, q) \leqslant \frac{1}{2} D_\sigma(W_{pq}^r, L_q) \leqslant (W_{pq}^r, L_q) \leqslant E_\sigma^L(W_{pq}^r, L_q) \leqslant \sigma^{-r}\tilde{\lambda}(r, p, q) + o(\sigma^{-r})$ (*if* $\sigma \geqslant 1$).

Proof. From the definition of the optimal intrinsic error and the optimal linear error and (2.16), we have

$$\begin{aligned} E_n(W_{pq}^r, L_q) &\leqslant E_n^L(W_{pq}^r, L_q) \\ &\leqslant \sup_{f \in W_{pq}^r} \|f(\cdot) - s_{n,r-1}(f, \cdot)\|_q \leqslant n^{-r}\tilde{\lambda}(r, p, q). \end{aligned} \tag{4.6}$$

Let $c_j = \int_j^{j+1} |f(t)|^{p'} dt$, $\forall j \in \mathbf{Z}$. Then by Hölder's inequality, we have

$$\begin{aligned} \|f\|_{L_{p'}[-N,N]} &= \left(\sum_{j=-N}^{N} c_j\right)^{1/p'} \leqslant \left(\sum_{j=-N}^{N} c_j^{q'/p'}\right)^{1/q'} (2N)^{1/p' - 1/q'} \\ &= (2N)^{1/p' - 1/q'} \|f\|_{L_{p'q'}(I_N)}. \end{aligned} \tag{4.7}$$

According to Lemmas 4.1 and 3.1(2) and (4.7) and by changing scale, for fixed $\xi \in \Theta_\sigma$, we have

$$\begin{aligned}
&\sup\{\|f\|_q : f \in W^r_{pq}, f(\xi_j)=0, j \in \mathbf{Z}\} \\
&\quad \geqslant E(W^r_{q'}, S_{r-1}(\xi) \cap L_{p'q'}, L_{p'q'}) \\
&\quad \geqslant E(W^{r,0}_{q'}(I_N), S_{r-1}(\xi)|_{I_N}, L_{p'q'}(I_N)) \\
&\quad \geqslant (2N)^{1/q'-1/p'} E(W^{r,0}_{q'}(I_N), S_{r-1}(\xi)|_{I_N}, L_{p'}(I_N)) \\
&\quad \geqslant (2N)^{1/q'-1/p'} d_{N(\xi)+r+2}(\tilde{W}^r_{q'}(I_N), L_{p'}(I_N)) \\
&\quad = (2N)^{1/q'-1/p'}(2N)^{r+1/p'-1/q'} d_{N(\xi)+r+2}(\tilde{W}^r_{q'}[0,1], L_{p'}[0,1]) \\
&\quad \geqslant (2N)^r \left(\left[\frac{N(\xi)+r+2}{2} \right] + 1 \right)^{-r} \lambda(r, q', p'), \qquad (4.8)
\end{aligned}$$

where $N(\xi) =: \operatorname{card}\{\xi \cap [-N, N]\} + r$, since $\xi = \{\xi_j\}_{j \in \mathbf{Z}} \in \Theta_\sigma$, i.e.,

$$\varliminf_{N \to +\infty} \frac{N(\xi)+r+2}{2N} \leqslant \sigma,$$

Then, from the fact that

$$D_\sigma(W^r_{pq}, L_q) = \inf_{\xi \in \Theta_\sigma} \sup_{f \in W^r_{pq}} \{\|f\|_q : f \in W^r_{pq}, I_\xi f = 0\}$$

(see [16]), we have

$$D_\sigma(W^r_{pq}, L_q) \geqslant \sigma^{-r} 2^r \lambda(r, q', p') = \sigma^{-r} 2^r \lambda(r, p, q) = \sigma^{-r} \tilde{\lambda}(r, p, q). \qquad (4.9)$$

On the other hand, by (4.5) (when $\mathfrak{M} = W^r_{pq}$) and Theorem 1, we have

$$\begin{aligned}
D_n(W^r_{pq}, L_q) &\leqslant E_n(W^r_{pq}, L_q) \leqslant E^L_n(W^r_{pq}, L_q) \\
&\leqslant \sup_{f \in W^r_{pq}} \|f - s_{n,r-1}(f)\|_q \leqslant n^{-r} \tilde{\lambda}(r, p, q). \qquad (4.10)
\end{aligned}$$

Thus, Theorem 5(1) follows from (4.9) and (4.10) for $\sigma = n \in Z_+$, and Theorem 5(2) follows from (4.5), (4.9), and Theorem 5(1).

5. Dual Case

Proof of Theorem 2. Similar to the proof of (2.3), we may verify that

$$\sup_{f \in W^r_p} \|f - s_{n,r-1}(f)\|_{qp} \leqslant n^{-r} \tilde{\lambda}(r, p, q), \qquad n \in Z_+ \qquad (5.1)$$

(for details of the proof of (5.1) readers may refer to [11] and the proof of (2.3)).

To get the lower estimate for $\bar{d}_n(W_p^r, L_{qp})$ in Theorem 2, we use the following

LEMMA 5.1. *Let $f_k(x, z)$ be as defined in* (3.4). *Set*

$$F_k(x, z) =: (2N)^{r-1/p} f_k\left(\frac{x+N}{2N}, z\right), \qquad x \in I_N, \forall N \geqslant 1, k \in \mathbf{Z}_+.$$

Then for any subspace $B \subset L_{qp}(I_N)$ with $\dim B = n$, *there is a $\hat{z} \in S^{n+r+1}$ such that $F_k(t, \hat{z}) \in W_p^{r,0}(I_N)$ and*

$$E(F_k(\cdot, \hat{z}), B, L_{qp}(I_N)) = \|F_k(\cdot, \hat{z})\|_{L_{qp}(I_N)}, \tag{5.2}$$

where $1 < q \leqslant p < \infty$.

Proof. It is easy to verify that (5.2) follows from Borsuk's theorem. We omit its details.

We now prove the lower estimate for $\bar{d}_\sigma(W_p^r, L_{qp})$ for any $\sigma \geqslant 1$. Let M be a subspace of $L_{qp}(R)$ of $\overline{\dim}\ M \leqslant \sigma$. For each $N \geqslant 1$, we take a linear subspace B of $L_{qp}(I_N)$ of $\dim B = K(\varepsilon, N, M) =: N_\sigma$ satisfying $E(S(M)_n, B, L_{qp}(I_N)) < \varepsilon$.

Then, by Lemma 5.1 and the inequality

$$\|F_k(\cdot, \hat{z})\|_{L_{qp}(I_N)} \geqslant (2N)^r \|f_k(\cdot, \hat{z})\|_{L_q[0,1]},$$

we have

$$\begin{aligned}
E(W_p^r, M, L_{qp}) &\geqslant E(W_p^{r,0}(I_N), M|_{I_N}, L_{qp}(I_N)) \\
&\geqslant \inf\{\|F_k(\cdot, \hat{z}) - f\|_{L_{qp}(I_N)}; f \in M|_{I_N}, \|f\|_{qp} \leqslant 2\|F_k(\cdot, \hat{z})\|_{qp}\} \\
&\geqslant \inf\{\|F_k(\cdot, \hat{z}) - g\|_{L_{qp}(I_N)}; g \in B\} \\
&\quad - 2\|F_k(\cdot, \hat{z})\|_{qp} E(S(M)_N, B, L_{qp}(I_N)) \\
&\geqslant (1-2\varepsilon)\|F_k(\cdot, \hat{z})\|_{qp} = (1-2\varepsilon)(2N)^r \|f_k(\cdot, \hat{z})\|_{L_q[0,1]} \\
&\geqslant (1-2\varepsilon)(2N)^r \min\{\|f_k(\cdot, z)\|_{L_q[0,1]}; z \in S^{N_\sigma+r+1}\}.
\end{aligned} \tag{5.3}$$

Letting $k \to \infty$, we have

$$\begin{aligned}
&E(W_p^r, M, L_{qp}) \\
&\quad \geqslant (1-2\varepsilon)(2N)^r \min\{\|f(\cdot, z)\|_{L_q[0,1]}; z \in S^{N_\sigma+r+1}\} \\
&\quad \geqslant (1-2\varepsilon)(2N)^r d_{2[(N_\sigma+r+2)/2]}(\tilde{W}_p^r[0, 1], L_q[0, 1]) \\
&\quad \geqslant (1-2\varepsilon)(2N)^r \left(2\left[\frac{N_\sigma+r+2}{2}\right]\right)^{-r} \tilde{\lambda}(r, p, q).
\end{aligned} \tag{5.4}$$

Letting $N \to \infty$, and $\varepsilon \to 0^+$, we have

$$E(W_p^r, M, L_{qp}) \geqslant \sigma^{-r}\tilde{\lambda}(r, p, q). \tag{5.5}$$

Thus, we have

$$\bar{d}_\sigma(W_p^r, L_{qp}) \geqslant \sigma^{-r}\tilde{\lambda}(r, p, q). \tag{5.6}$$

Theorem 2 follows immediately from (5.1) and (5.6) for $\sigma = n \in Z_+$. Analogous to Theorems 4 and 5, we have:

THEOREM 6. *Let* $1 < q \leqslant p < \infty$. *Then*

(1) $\tilde{d}_n(W_p^r, L_{qp}) = \inf_{f \in W_p^r} \|f(\cdot) - s_{n,r-1}(f, \cdot)\|_{qp} = n^{-r}\tilde{\lambda}(r, p, q)$ $(n \in \mathbf{Z}_+)$,

(2) $\tilde{d}_\sigma(W_p^r, L_{qp}) = \sigma^r\tilde{\lambda}(r, p, q) + o(\sigma^{-r})$ $(\sigma \geqslant 1)$.

THEOREM 7. *Let* $1 < q \leqslant p < \infty$. *Then*

(1) $\frac{1}{2}D_n(W_p^r, L_{qp}) = E_n(W_p^r, L_{qp}) = E_n^L(W_p^r, L_{qp}) = \sup_{f \in W_p^r} \|f(\cdot) - s_{n,r-1}(f, \cdot)\|_{qp} = n^{-r}\tilde{\lambda}(r, p, q)$ $(n \in \mathbf{Z}_+)$.

(2) $\sigma^{-r}\tilde{\lambda}(r, p, q) \leqslant \frac{1}{2}D_\sigma(W_p^r, L_{qp}) \leqslant E_\sigma(W_p^r, L_{qp}) \leqslant E_\sigma^L(W_p^r, L_{qp}) \leqslant \sigma^{-r}\tilde{\lambda}(r, p, q) + o(\sigma^{-r})$ $(\sigma \geqslant 1)$.

Remark 5.1. If we denote

$$\tilde{\lambda}(r, p, q) = \tilde{\lambda}(r, q', p') =: \|E\|_{L_q[0,1]}, \qquad 1 \leqslant q < p = \infty,$$

where $E(x)$ is the Euler spline of degree r with period 2 (see [14]), then it is easy to see that Theorems 2, 6, and 7 are also valid in the cases $1 = q < p \leqslant \infty$ and $1 \leqslant q < p = \infty$.

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